On "non-Hermitian Quantum Mechanics".

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A series of recent papers *"Faster than Hermitian Quantum Mechanics"* [1] and related articles such as [2, 3] made a point of the possibility of a non-Hermitian, but PT-symmetric, operator to play the role of a Hamiltonian. In particular, they show that with an appropriate choice of an inner product, the evolution generated by such an operator will conserve the norm and scalar product. Before proceeding to our main observation we would first like to note that this choice of appropriate product depends on the spectrum of the Hamiltonian. We feel that this is problematic as it does not lend itself well to a general theory. Furthermore, if one is to consider time dependant Hamiltonians, this might lead to a theory where the inner product itself is not only dependant on the Hamiltonian but also on time. Moreover, here we would like to show that if one chooses such an inner product then the Hamiltonian in question is actually Hermitian, and the whole exercise is to a certain degree redundant.

An operator is termed Hermitian or self adjoint if $A^\dagger = A$, that is $\langle Ax \, | y \rangle = \langle x \, | Ay \rangle$ for all $x, y \in \mathcal{H}$ [4, 5]. Therefore, Hermiticity of an operator is not an intrinsic property of the operator itself, but rather a property of the operator with reference to the Hilbert Space on whose elements it operates and, in particular, the associated inner product. For instance, with conventional (dot) inner product, in the Heisenberg matrix formulation of quantum mechanics, $A^\dagger$ is defined as matrix transposition and complex conjugate, $A = (A^*)^T$, of the operator A. While the inner product considered in [1, 2] is not defined in this way and, in general, depends on the Hamiltonian itself. Hence $A^\dagger$, and concomitantly Hermiticity, is defined with respect to this new inner product. The claim that PT-symmetric Hamiltonian is not Hermitian because it does not satisfy the Hermiticity condition with respect to the standard (dot) inner product is therefore not consistent with the choice of Hilbert space. As we will show below, it must be a Hermitian (in the general sense of Hermiticity) in order to have a consistent quantum mechanical theory with conserved norm.

Let an operator H generate the evolution through the Schrödinger equation, $i\hbar\frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle$. By multiplying on both sides by $\langle \phi |$ we get $i\hbar \left\langle \phi \left| \frac{\partial}{\partial t}\psi \right.\right\rangle = \langle \phi \, | H\psi \rangle$. Similarly we find $i\hbar \left\langle \psi \left| \frac{\partial}{\partial t}\phi \right.\right\rangle = \langle \psi \, | H\phi \rangle$. Taking complex conjugate of the second equation and subtracting it from the first we find $i\hbar \left[\left\langle \phi \left| \frac{\partial}{\partial t}\psi \right.\right\rangle + \left\langle \frac{\partial}{\partial t}\phi \, | \psi \right\rangle \right] = \langle \phi \, | H\psi \rangle - \langle H\phi \, | \psi \rangle$. Now we recognise the left hand side as a total derivative with respect to time, and use the definition of the Hermitian conjugate, which yields

$$i\hbar \frac{\partial}{\partial t} \langle \phi \, | \psi \rangle = \langle \phi \, | H\psi \rangle - \left\langle \phi \left| H^\dagger \psi \right.\right\rangle = \left\langle \phi \left| \left(H - H^\dagger \right) \psi \right.\right\rangle . \tag{1}$$

It follows from (1) that if H is Hermitian, then the norm and the scalar product are conserved. On the other hand, if the scalar product is conserved for any two functions from a full basis of the Hilbert space, the Hamiltonian will be Hermitian.

In some textbooks such as [6], normalisation together with Schrödinger evolution is taken as a sufficient condition that the Hamiltonian be Hermitian, which is not the case. The preservation of the norm only fixes the diagonal matrix elements of the operator H, $\left\langle \psi \left| \left(H - H^\dagger \right) \psi \right.\right\rangle$.

Hence, the structure of the Hilbert space imposed by introducing a norm preserving inner product as introduced in [1, 2] implies the Hamiltonian is actually Hermitian. An interesting corollary of this is that by changing the inner product on a Hilbert space we also change the set comprising all Hermitian operators acting on this space. As only members of this set can represent observable we note that the ramifications of changing the inner product extend beyond simply determining the Hermiticity of the Hamiltonian. In this respect, one concludes that what is called as PT symmetric quantum mechanics is just the ordinary quantum mechanic with different (more involved and possibly problematic) inner product. Therefore, this type of analysis can not be considered as a non-Hermitian extension or generalisation of quantum mechanics.

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